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ON EQUI-ASYMPTOTIC STABILITY WITH RESPECT TO PART OF THE VARIABLES†

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A system of equations of perturbed motion in which the right-hand sides are almost periodic functions of time is considered. A sufficient condition for the trivial solution of the system to be equi-asymptotically stable with respect to part of the variables is proved. © 2000 Elsevier Science Ltd. All rights reserved.

The main method for investigating the stability and asymptotic stability of the solution $x = 0$ of a system of differential equations of perturbed motion

$$\dot{x} = X(x,t); \quad x = (x_1,\ldots,x_n), \quad X = (X_1,\ldots,X_n) \tag{1}$$

with respect to part of the variables is Lyapunov's Second Method. It is based on the construction of a Lyapunov function $V(x, t)$.

Rumyantsev [1] proved a basic theorem according to which the solution $x = 0$ of system (1) is stable with respect to part of the variables, which is an analogue of Lyapunov's stability theorem, on the assumption that the function $V(x, t)$ is y-positive-definite and its derivative along trajectories of Eqs (1) satisfies the condition $dV/dt \leqslant 0$. Later [2, 3] a theorem was proved stating that the trivial solution of system (1) is asymptotically stable with respect to part of the variables, on the assumption that the derivative dV/dt along trajectories of Eqs (1) is y-negative-definite.

In applied problems one is frequently able to construct a y-positive-definite function $V(x, y)$ whose derivative dV/dt along trajectories of Eqs (1) is only non-positive. Under those conditions, one can prove [4, 5] that the trivial solution of an autonomous system of equations of perturbed motion

$$\dot{x} = X(x) \tag{2}$$

is asymptotically stable.

It has been shown [6] that the analogous theorem is not true for the general case of a non-autonomous system.

In this note we will consider a more general case than that of an autonomous system: on the assumption that the right-hand sides of system (2) are almost periodic functions of it, will be proved that the solution $x = 0$ is equi-asymptotically y-stable (that is, $y(t) \to 0$ as $t \to \infty$ uniformly with respect to the initial perturbations x_0).

By analogy with previously introduced notation [7], we let $x_1, \ldots, x_m (m > 0, n = m + p, p \geqslant 0)$ denote the variables with respect to which the stability of the solution $x = 0$ of system (1) is being investigated. For convenience, we will denote these variables by $y_i = x_i$ $(i = 1, \ldots, m)$ and the other variables by $z_j = x_{m+j}$ $(j = 1, \ldots, p)$, that is, we express the vectors x and X in the form

$$x = (y_1,\ldots,y_m,\ z_1,\ldots,z_p)^T \equiv (y,z)^T$$

$$X = (Y_1,\ldots,Y_m,\ Z_1,\ldots,Z_p)^T \equiv (Y,Z)^T$$

We also put

$$\|y\| = \left(\sum_{i=1}^{m} y_i^2\right)^{1/2}, \quad \|z\| = \left(\sum_{j=1}^{p} z_j^2\right)^{1/2}, \quad \|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = (\|y\|^2 + \|z\|^2)^{1/2}$$

We will introduce a few definitions.

Definition 1 [8]. A continuous function $f(t)$ with values in R^n will be called uniformly almost periodic if, for any $\varepsilon > 0$ and every $r > 0$, $L = L(\varepsilon, r)$ exists such that, in any interval $[\alpha, \alpha + L(\varepsilon)]$, $\alpha \in (-\infty; +\infty)$ there is at least one number τ for which

$$\|f(t) - f(t+\tau)\| < \varepsilon, \quad -\infty < t < +\infty$$

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Definition 2 [9]. A continuous function $f(x, t)$ ($x \in R^s$, $-\infty < t < +\infty$), with values in R^n will be called uniformly almost periodic if, for any $\varepsilon > 0$ and every $r > 0$, $L = L(\varepsilon, r)$ exist such that, in any interval $[\alpha, \alpha + L(\varepsilon, r)]$, $\alpha \in (-\infty; +\infty)$, there is at least one number τ for which

$$\|f(x,t)-f(x,t+\tau)\|<\varepsilon, \quad -\infty<t<+\infty, \quad \|x\|<r$$

Definition 3 [7]. The motion $x = 0$ is said to be equi-asymptotically y-stable if, for every $t_0 \geqslant 0$, $\delta(t_0) > 0$ exists such that $\|y(t; t_0, x_0)\| \to 0$ uniformly in $\|x_0\| < \delta(t_0)$ as $t \to \infty$; that is, for any $\varepsilon > 0$, $T(\varepsilon, t_0) > 0$ exists such that $\|x_0\| < \delta$ implies $\|y(t; t_0, x_0)\| < \varepsilon$ for all $t \geqslant t_0 + T$.

Consider the equations of perturbed motion (1). The function $X(x, t)$ is assumed to be defined, continuous and Lipschitzian with respect to x in the domain

$$t \in R, \quad \|y\|<H, \quad \|z\|<+\infty, \quad H=\text{const} \tag{3}$$

It will also be assumed that the solutions of system (1) are z-continuable. This means [7] that any solution $x(t)$ is defined for all $t \geqslant 0$ such that $\|y(t)\| \leqslant H$.

Theorem. Suppose the equations of perturbed motion (1) are such that

1. every solution of system (1) that begins in a neighbourhood of the point $x = 0$ is bounded;
2. one can construct a function $V(x, t)$ which is almost periodic in t, y-positive-definite, continuously differentiable and satisfies the inequality $dV/dt \leqslant 0$ in domain (3); moreover, the derivative dV/dt may vanish only at points of a set that does not contain an entire semi-trajectory $x(x_0, t_0, t)$, ($t_0 < t < +\infty$) of system (1) (not counting the trivial solution).

Then the solution $x = 0$ is equi-asymptotically y-stable.

Before proving the theorem, we will formulate a few auxiliary propositions.

Lemma 1. The functions $X(x, t)$ and $V(x, t)$ are uniformly almost periodic.

This lemma was proved in [9].

Lemma 2. For any $\varepsilon > 0$ an unbounded increasing sequence of ε-almost-periods $\{\tau_i\}$ exists, common to the functions $X(x, t)$ and $V(x, t)$

$$\|X(x,t)-X(x,t+\tau_i)\|<\varepsilon, \quad |V(x,t)-V(x,t+\tau_i)|<\varepsilon$$

The proof follows from Kronecker's theorem [10].

Lemma 3. Let $x(x_0, t_0, t)$, ($t_0 < t < +\infty$) be a semi-trajectory of system (1) satisfying the initial condition $x(x_0, t_0, t) = x_0$ and contained in domain (3); let $\{\varepsilon_k\}$ be a sequence of positive numbers converging monotonically to zero, and let $\{\tau_k\}$ be some sequence of ε_k-almost-periods of the vector function $X(x, t)$ (each ε_k is associated with the ε_k-almost-period τ_k), where $\{\tau_k\}$ is monotone increasing and $\tau_k \to \infty$ as $k \to \infty$. Then

$$\lim_{k\to\infty}\|x(x_k,t_0,t^*)-x(x_0,t_0,t^*+\tau_k)\|=0, \quad x_k=x(x_0,t_0,t_0+\tau_k) \tag{4}$$

where t^* is some time greater than t_0.

The proof may be found in [11].

To prove the theorem, we will use a method described in [11, 12].

We first note that y-stability of the trivial solution follows from Rumyantsev's theorem [1].

It can be shown that $\|y(x_0, t_0, t)\| \to 0$ as $\tau \to \infty$.

Suppose the contrary. The function $V(x(x_0, t_0, t), t)$ is not monotone increasing, since $dV/dt \leqslant 0$. Hence the limit $\lim_{t\to\infty} V(x(x_0, t_0, t), t) = V_0$ exists and $V(x(x_0, t_0, t), t) \geqslant V_0$ for any $t > t_0$. It follows from our assumption that $V_0 \neq 0$. Now let $\{\varepsilon_i\}$ be a sequence of positive numbers converging monotonically to zero. For any ε_i a sequence of almost-periods $\tau_{i1}, \tau_{i2}, \ldots, \tau_{in}$ for the functions $V(x, t)$ and $X(x, t)$ exists, which converges to infinity. We can write

$$|V(x,t)-V(x,t+\tau_{in})|<\varepsilon_i, \quad \|X(x,t)-X(x,t+\tau_{in})\|<\varepsilon_i$$

$$\|x\|\leqslant\varepsilon, \quad -\infty<t<+\infty$$

Let us assume that $\tau_{in} < \tau_{i+1,n}$, putting $\tau_{kk} = \tau_k$. Consider the sequence of points $x_k = x(x_0, t_0, t_0 + \tau_k)$ ($k = 1, 2, \ldots$). By Condition 1 of the theorem, this sequence is bounded. Hence we can extract a convergent subsequence. For simplicity, we may assume that the sequence $\{x_k\}$ itself is convergent. Let x^* be a limit point of the sequence $\{x_k\}_{k=1}^{\infty}$. It follows from our assumption that $x^* \neq 0$. Using the fact that $V(x, t)$ is continuous and almost periodic, we can write

$$V(x^*,t_0)=\lim_{k\to\infty} V(x_k,t_0+\tau_k)=\lim_{k\to\infty} V(x(x_0,t_0,t_0+\tau_k),\ t_0+\tau_k)=V_0$$

Consider the semi-trajectory $x(x^*, t_0, t)(t_0 < t < \infty)$. By assumption, it must contain points at which $dV(x(x^*, t_0, t), t)/dt < 0$, that is, one can find $t^* > t_0$ such that $V(x(x^*, t_0, t^*), t^*) = V_1 < V_0$.

By the continuity of the solutions as functions of the initial data, we have

$$x(x^*, t_0, t^*) = \lim_{k\to\infty} x(x_k, t_0, t^*)$$

Consequently

$$\lim_{k\to\infty} V(x(x_k, t_0, t^*), t^*) = V_1 \tag{5}$$

Since $X(x, t)$ is almost periodic and condition (4) is true, we obtain

$$\|x(x_k, t_0, t^*) - x(x_0, t_0, t^* + \tau_k)\| \leq \gamma_k, \quad \lim_{k\to\infty} \gamma_k = 0 \tag{6}$$

The fact that $V(x, t)$ is uniformly almost periodic implies that

$$|V(x, t^*) - V(x, t^* + \tau_k)| < \varepsilon_k \tag{7}$$

It follows from (5) and (6) that

$$|V(x(x_0, t_0, t^* + \tau_k), t^*) - V_1| < \eta_k, \quad \lim_{k\to\infty} \eta_k = 0 \tag{8}$$

Adding inequality (7) for $x = (x(x_0, t_0, t^* + \tau_k))$ and inequality (8), we obtain

$$|V(x(x_0, t_0, t^* + \tau_k), t^* + \tau_k) - V_1| < \eta_k + \varepsilon_k; \quad \eta_k + \varepsilon_k \to 0 \quad \text{as} \quad k \to \infty \tag{9}$$

But

$$\lim_{k\to\infty} V(x(x_0, t_0, t^* + \tau_k),\ t^* + \tau_k) = V_0 \tag{10}$$

Relations (9) and (10) contradict one another, since $V_1 < V_0$. Consequently, our assumption that $V_0 \neq 0$ is false, so $V_0 = 0$. Since $V(x, t)$ is y-positive-definite, it follows that

$$a(\|y\|) \leq V(x, t) \tag{11}$$

where a is some Hahn function [13]. Thus, $V(t, x) \to 0$ as $t \to \infty$. Using estimate (11), we conclude that $a(\|y(x_0, t_0, t)\|) \to 0$ as $t \to \infty$. Consequently, $y(x_0, t_0, t)\|) \to 0$.

By the assumption of the theorem, $V(x, t) \geq a(\|y(x_0, t_0, t)\|)$. We have already proved that $\lim_{t\to\infty} V(x(x_0, t_0, t), t) = 0$. The derivation of the limit relationship $\|y(x_0, t_0, t)\| \to 0$ as $t \to \infty$ uniformly in x_0 in a δ-neighbourhood of the origin follows from a previous result [7], the number $\delta = \delta(\varepsilon, t_0)$ being determined from the y-stability condition: $\|x_0\| \leq \delta \to \|y(t)\| < \varepsilon$ for any $t > t_0$. This proves that the trivial solution of system (1) is indeed equi-asymptotically y-stable.

Remark. An analogous theorem may be proved for quasi-periodic systems, which constitute a special case of almost periodic systems, by the method of limit systems, using Theorem 2.2 of [14] and the results of [15].

Example. Consider the system of equations

$$\frac{dz}{dt} = z\sin(y_1^2 + y_2^2)(\sin t + \sin\sqrt{3}t) - z^3$$

$$\frac{dy_1}{dt} = -y_2^3 - y_1(y_2^2 + 1)(1 + z^2 + \sin^2 2t + \cos^2\sqrt{2}t)$$

$$dy_2/dt = y_1^3$$

Take $V = (y_1^4 + y_2^4)/4$. Then

$$dV/dt = -y_1^4(y_2^2 + 1)(1 + z^2 + \sin^2 2t + \cos^2\sqrt{2}t) \leq 0$$

dV/dt may vanish only in the set $y_1 = 0$, which does not include whole semi-trajectories of the system. Consequently, by the theorem just proved, the trivial solution of our system is equi-asymptotically stable with respect to y_1, y_2.

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